

Dynamics of Rotating Cylindrical Shells in General Relativity

Paulo R.C.T. Pereira *

Departamento de Astrofísica, Observatório Nacional – CNPq, Rua General José Cristino 77, São Cristóvão, 20921-400 Rio de Janeiro – RJ, Brazil

Anzhong Wang [†]

Departamento de Física Teórica, Universidade do Estado do Rio de Janeiro, Rua São Francisco Xavier 524, Maracanã, 20550-013 Rio de Janeiro – RJ, Brazil
(February 7, 2008)

Cylindrical spacetimes with rotation are studied using the Newmann-Penrose formulas. By studying null geodesic deviations the physical meaning of each component of the Riemann tensor is given. These spacetimes are further extended to include rotating dynamic shells, and the general expression of the surface energy-momentum tensor of the shells is given in terms of the discontinuation of the first derivatives of the metric coefficients. As an application of the developed formulas, a stationary shell that generates the Lewis solutions, which represent the most general vacuum cylindrical solutions of the Einstein field equations with rotation, is studied by assuming that the spacetime inside the shell is flat. It is shown that the shell can satisfy all the energy conditions by properly choosing the parameters appearing in the model, provided that $0 \leq \sigma \leq 1$, where σ is related to the mass per unit length of the shell.

PACS numbers: 04.20Cv, 04.30.+x, 97.60.Sm, 97.60.Lf.

I. INTRODUCTION

Gravitational collapse of a realistic body has been one of the most thorny and important problems in Einstein's theory of General Relativity. Due to the complexity of the Einstein field equations, the problem even in simple cases, such as, spacetimes with spherical symmetry, is still not well understood [1], and new phenomena keep emerging [2]. In 1991, Shapiro and Teukolsky [3] studied numerically the problem of a dust spheroid, and found that only the spheroid is compact enough, a black hole can be formed. Otherwise, the collapse most likely ends with a naked singularity. Since then, the gravitational collapse with non-spherical symmetry has attracted much attention. In particular, by studying the collapse of a cylindrical shell that consists of counter-rotating particles, Apostolatos and Thorne (AT) found that the rotation always halts the collapse [4]. As a result, no naked singularities can be formed on the symmetry axis. However, in the AT work only the case where the total angular momentum of the collapsing shell is zero was considered. In more realistic case, the spacetime has neither cylindrical symmetry nor zero angular momentum. As a generalization of the AT work, in this paper we shall consider the case where the total angular momentum is not zero, while still keep the requirement that the spacetime be cylindrical.

Another motivation for us to consider rotating shell comes from recent study of the physical interpretation of the Levi-Civita vacuum solutions [5], and the Levi-Civita solutions with cosmological constant (LCC) [6]. These solutions have been known for a long time [7], but their physical properties were studied extensively only very recently [8], and there are still several open problems to be solved. By looking for some physical sources to the LC solutions, we extended one of the two parameter that is related to, but not equal to, the mass per unit length, from the range, $[0, \frac{1}{4}]$, to the range, $[0, 1]$ [5]. However, beyond this range, the physical meaning of the solutions is still not clear. Despite of the simplicity of the solutions, it was found that they have very rich physical meaning. In particular, they can give rise to black hole structures [6]. Since the LC and LCC solutions are all static, it is very interesting to generalize these studies to the rotating case.

As it can be seen from the discussions given below, when rotation is included, the problem is considerably complicated. This partially explains why spacetimes with rotation are hardly studied (analytically). Thus, to start with, in the next section (Sec. II) we shall first study the main properties of cylindrical spacetimes with rotations, using the Newmann-Penrose (NP) formulas [9]. One of the main reasons to use the NP formulas is that they give directly physical interpretation for each component of the Riemann tensor. This is particularly useful when spacetimes contain

*Email: terra@maxwell.on.br

[†]Email: wang@dft.if.uerj.br

gravitational waves. In Sec. III, the spacetimes are extended to include the case where rotating matter shells appear. To deal with the problem, one usually uses Israel's method [10]. However, we find that for the present problem Israel's method becomes very complicated and is very difficult to be implemented. Instead, we shall follow Darmois [11] and Lichnerwicz [12] (see also Papapetrou and Hamoui [13] and Taub [14]). Although the two methods are essentially equivalent [15], the latter is simpler, specially in dealing with complicate boundaries like the present one. The disadvantage of the latter is that it requires only one set of coordinates across the boundaries, while Israel's method does not. In this section, the physical interpretation of the surface EMT is also studied by solving the corresponding eigenvalue problem. As an application of the developed formulas, in Sec. IV we consider a stationary shell that generates the Lewis solutions, which represent the most general cylindrical vacuum solutions of the Einstein field equations with rotation [16], while in Sec. V our main conclusions are given.

II. CYLINDRICAL SPACETIMES WITH ROTATION

To begin with, let us consider the cylindrical spacetimes with rotation described by the metric [17],

$$ds^2 = e^{2(\chi-\psi)} (dt^2 - dr^2) - W^2 e^{-2\psi} (\omega dt + d\varphi)^2 - e^{2\psi} dz^2, \quad (1)$$

where ψ , χ , W and ω are functions t and r , and $\{x^\mu\} \equiv \{t, r, z, \varphi\}$, ($\mu = 0, 1, 2, 3$), are the usual cylindrical coordinates. In general the spacetimes have two Killing vectors, one is associated with the invariant translations along the symmetry axis, $\xi_{(z)} = \partial z$, and the other is associated with the invariant rotations about the axis, $\xi_{(\varphi)} = \partial \varphi$. Clearly, for the metric given above, the two Killing vectors are orthogonal. When $\omega = 0$, the metric represents spacetimes without rotation, in which the polarization of gravitational waves has only one degree of freedom and the direction of polarization is fixed [18,19]. For the spacetimes to be cylindrical, several criteria have to be satisfied [20]. When the symmetry axis is regular, those conditions are easily imposed. However, when it is singular, it is still not clear which conditions should be imposed [21].

Corresponding to the metric (1) the Christoffel symbols and Einstein tensor are given, respectively, by Eqs.(A.2) and (A.3). To further study the spacetimes, let us consider the Ricci and Weyl scalars, as these quantities have their explicit physical interpretations [9], and they are particularly useful in the study of gravitational interaction among mater fields and gravitational waves [22,23]. Choosing the null tetrad ¹

$$\begin{aligned} l^\mu &= \frac{e^{\psi-\chi}}{\sqrt{2}A} \{1, 1, 0, -\omega\}, \quad n^\mu = \frac{Ae^{\psi-\chi}}{\sqrt{2}} \{1, -1, 0, -\omega\}, \\ m^\mu &= \frac{1}{\sqrt{2}} \left\{ 0, 0, e^{-\psi}, i \frac{e^\psi}{W} \right\}, \quad \bar{m}^\mu = \frac{1}{\sqrt{2}} \left\{ 0, 0, e^{-\psi}, -i \frac{e^\psi}{W} \right\}, \end{aligned} \quad (2)$$

where a bar denotes the complex conjugate, and A is an arbitrary function of t and r , we find that the spin coefficients are given by Eq.(B.1). Since $\kappa = \nu = 0$, the two null vectors l^μ and n^μ define two null geodesic congruences [25]. The one defined by l_μ is outgoing, while the one defined by n_μ is ingoing, similar to the no-rotating case [24]. In fact, it is easy to show that

$$l_{\mu;\nu} l^\nu = 2\epsilon l_\mu, \quad n_{\mu;\nu} n^\nu = 2\gamma n_\mu, \quad (3)$$

where ϵ and γ are given by Eq.(B.1). Choosing the function A as $A = e^{\chi-\psi}$, from Eq.(B.1) we find that $\epsilon = 0$, and then Eq.(3) shows that the null geodesic congruence defined by l_μ is affinely parameterized. Hence, the corresponding expansion, rotation and shear of the outgoing null geodesic congruence are given, respectively, by

$$\begin{aligned} \theta_l &\equiv \frac{1}{2} l^\mu_{;\mu} = -Re(\rho) = \frac{e^{2(\psi-\chi)}}{2\sqrt{2}} \frac{W_{,t} + W_{,r}}{W}, \\ \omega_l^2 &\equiv \frac{1}{2} l_{[\mu;\nu]} l^{\mu;\nu} = [Im(\rho)]^2 = 0, \\ \sigma_l &\equiv \left(\frac{1}{2} l_{(\mu;\nu)} l^{\mu;\nu} - \theta_l^2 \right)^{1/2} = \frac{e^{2(\psi-\chi)}}{2\sqrt{2}} \left[2(\psi_{,t} + \psi_{,r}) - \frac{W_{,t} + W_{,r}}{W} \right], \end{aligned} \quad (4)$$

¹It should be noted that the choice of the null tetrad used here is different from the one used in [24] for the case $\omega = 0$, and can be obtained one from the other by exchanging the roles of the two null vectors, l_μ and n_μ .

where

$$A_{(\mu\nu)} \equiv \frac{1}{2} (A_{\mu\nu} + A_{\nu\mu}), \quad A_{[\mu\nu]} \equiv \frac{1}{2} (A_{\mu\nu} - A_{\nu\mu}), \quad (5)$$

and $W_{,t} \equiv \partial W / \partial t$, $W_{,r} \equiv \partial W / \partial r$, etc. Similarly, if we choose the arbitrary function A as $A = e^{\psi-\chi}$, Eq.(B.1) yields $\gamma = 0$, then Eq.(3) shows that the null geodesic congruence defined by n_μ now is affinely parameterized, and the corresponding expansion, rotation and shear of the ingoing null geodesic congruence are given, respectively, by

$$\begin{aligned} \theta_n &\equiv \frac{1}{2} n^\mu{}_{;\mu} = Re(\mu) = \frac{e^{2(\psi-\chi)}}{2\sqrt{2}} \frac{W_{,t} - W_{,r}}{W}, \\ \omega_n^2 &\equiv \frac{1}{2} n_{[\mu;\nu]} n^{\mu;\nu} = [Im(\mu)]^2 = 0, \\ \sigma_n &\equiv \left(\frac{1}{2} n_{(\mu;\nu)} n^{\mu;\nu} - \theta_n^2 \right)^{1/2} = \frac{e^{2(\psi-\chi)}}{2\sqrt{2}} \left[2(\psi_{,t} - \psi_{,r}) - \frac{W_{,t} - W_{,r}}{W} \right]. \end{aligned} \quad (6)$$

Once the spin coefficients are given, we can calculate the corresponding Ricci and Weyl scalars, which are given, respectively, by Eqs.(B.2) and (B.3). From those expressions we can see that all the Weyl scalars are non-zero, and each of them has the following physical interpretation [22,23]: The Ψ_0 and Ψ_1 terms represent, respectively, the transverse and longitudinal gravitational wave components along the null geodesic congruence defined by n^μ , and the Ψ_4 and Ψ_3 terms represent, respectively, the transverse and longitudinal gravitational wave components along the null geodesic congruence defined by l^μ , while the Ψ_2 term represents the ‘‘Coulomb’’ component.

The physical meaning of the Weyl and Ricci scalars can be further studied from geodesic deviations. Because of the symmetry, it is sufficient only to consider the null geodesics defined by l^μ , which are affinely parameterized when $A = e^{\chi-\psi}$, as shown above. Let η^μ be the deviation vector between two neighbor geodesics, and $\eta^\mu l_\mu = 0$. Then, using Eqs.(B.2) -(B.7), we find that the geodesic deviation can be written in the form

$$\begin{aligned} \frac{D^2 \eta^\mu}{D\lambda^2} &= -R^\mu_{\nu\alpha\beta} l^\nu l^\beta \eta^\alpha \\ &= \{ \Phi_{00} e_0^{\mu\nu} + \Psi_0 e_+^{\mu\nu} + i(\Psi_1 + \Phi_{01}) e_{03}^{\mu\nu} + i(\Psi_1 + \Phi_{01}) e_{13}^{\mu\nu} \} \eta_\nu, \end{aligned} \quad (7)$$

where Ψ_0 , Ψ_1 , Φ_{00} , and Φ_{01} are given by Eqs.(B.3) and (B.4) with $A = e^{\chi-\psi}$, and

$$\begin{aligned} e_0^{\mu\nu} &\equiv e_2^\mu e_2^\nu + e_3^\mu e_3^\nu, \quad e_+^{\mu\nu} \equiv e_2^\mu e_2^\nu - e_3^\mu e_3^\nu, \\ e_{03}^{\mu\nu} &\equiv e_0^\mu e_3^\nu + e_3^\mu e_0^\nu, \quad e_{13}^{\mu\nu} \equiv e_1^\mu e_3^\nu + e_3^\mu e_1^\nu, \end{aligned} \quad (8)$$

with

$$\begin{aligned} e_0^\mu &\equiv \frac{l^\mu + n^\mu}{\sqrt{2}}, \quad e_1^\mu \equiv \frac{l^\mu - n^\mu}{\sqrt{2}}, \\ e_2^\mu &\equiv \frac{m^\mu + \bar{m}^\mu}{\sqrt{2}}, \quad e_3^\mu \equiv -\frac{i(m^\mu - \bar{m}^\mu)}{\sqrt{2}}. \end{aligned} \quad (9)$$

Eqs.(7) has the following physical interpretation [22,23]. Let S_O and S_P be infinitesimal 2-elements spanned by e_2^μ and e_3^μ and orthogonal to a null geodesic C defined by l^μ , passing S_O and S_P at the points O and P , respectively. Let S be an infinitesimal circle with center O , lying in S_O as illustrated by Fig.1. Suppose that a light beam meets S_O in the circle S , then each of them has the following effect on the image of the circle S on S_P . The first term Φ_{00} in Eq.(7) will always make the circle contracted, as for any physically realistic matter field we have $\Phi_{00} \geq 0$ [cf. Fig.2(a)]. The second term Ψ_0 will make the circle elliptic with the main major axis along e_2^μ , as shown by Fig.2(b). To see the physical interpretation of the last two terms, let us consider a tube along the null geodesic C . Consider a sphere consisting photons, which will cut S_O in the circle S with the point O as its center, as shown in Fig.3. Then, the last term in Eq.(7) will make the image of the sphere at the point P as a spheroid with the main major axis along a line at 45° with respect to e_1^μ in the plane spanned by e_1^μ and e_3^μ , while the rays are left undeflected in the e_2^μ -direction. This can be seen clearly by performing a rotation in the plane spanned by e_1^μ and e_3^μ ,

$$e_1^\mu = \cos \alpha e_1'^\mu - \sin \alpha e_3'^\mu, \quad e_3^\mu = \sin \alpha e_1'^\mu + \cos \alpha e_3'^\mu, \quad (10)$$

which leads to

$$e_{13}^{\mu\nu} = \sin(2\alpha)(e_1'^\mu e_1'^\nu - e_3'^\mu e_3'^\nu) + \cos(2\alpha)(e_1'^\mu e_3'^\nu + e_3'^\mu e_1'^\nu). \quad (11)$$

Thus, choosing $\alpha = \pi/4$, we have

$$e_{13}^{\mu\nu} = e_1'^\mu e_1'^\nu - e_3'^\mu e_3'^\nu, \quad (\alpha = \pi/4). \quad (12)$$

Combining Eq.(7) and the above equation, we can see that the last term will make a circle in the $e_1' e_3'$ -plane into an ellipse with its main major axis along the e_1' -axis, which is at 45° with respect to the e_1 -axis. It should be noted that in the case of timelike geodesics the Ψ_1 term deflects the sphere into an ellipsoid [22]. Moreover, the third term in Eq.(7) is absent in the timelike geodesic case. The effect of this term will make a clock “flying” with the photons slow down, in addition to the effect of deflecting the photons in the e_3' -direction. It is interesting to note that there is a fundamental difference between the time delay caused by this term and the one caused by a Lorentz boost. The latter, in particular, has no contribution to geodesic deviations, timelike or null. From the above analysis we can see that for a pure Petrov type N gravitational wave propagating along the null geodesic congruence, in which only the component Ψ_0 is different from zero, the gravitational wave has only one polarization state, similar to the case without rotation [18,22,23]. The difference between these two cases is that in the case without rotation, the polarization angle remains the same even in different points along the wave path, while when $\omega \neq 0$, in general this is no longer true. In fact, it is easy to show that

$$e_{2;\nu}^\mu l^\nu = 0, \quad e_{3;\nu}^\mu l^\nu = -W e^{\psi-\chi} \frac{\omega, r}{2\sqrt{2}} (e_0^\mu + e_1^\mu). \quad (13)$$

Thus, although e_2^μ is parallel-transported along the null geodesic congruence, e_3^μ in general is not, and is rotating with respect to a parallel-transported basis. Since the polarization angle of the Ψ_0 wave remains the same with respect to e_3^μ , the polarization direction is also rotating with respect to the parallel-transported basis.

III. ROTATING CYLINDRICAL SHELLS

In the last section, the main properties of the spacetimes with rotation have been studied. Since the Einstein field equations are all involved with derivatives of the metric coefficients up to the second order, and the Bianchi identities, which are usually considered as representing the interaction among gravitational fields and matter fields, up to the third order, it is generally assumed that the metric coefficients are at least C^3 [27], that is, the derivatives of the metric coefficients exist and continuous at least up to their third order. However, for a long time it has been realized that this condition is too strict and rules out many physically interesting cases, such as, shells, star, and so on.

In this section, we shall generalize the formulas given in the last section to the case where the metric coefficients are C^3 only in certain regions, while across the boundaries that separate these regions they are C^0 , that is, the metric coefficients are only continuous across the boundaries. These boundaries can be classified into two different kinds, one is boundary surfaces, like a star, the other is surface layers, like a matter shell [10]. In the former case, the extrinsic curvatures of the boundaries in their two faces are equal, while in the latter case they are not, and as a result matter shells in general appear on these boundaries. In this paper, we shall treat the two cases together and consider the former is a particular case of the latter.

In a given spacetime, there may exist many disconnected boundaries. However, in the following we shall consider the case where there exists only one boundary, as its generalization to the cases of many boundaries is straightforward. Assume that the whole spacetime is divided into two regions V^\pm by a hypersurface Σ , where

$$V^+ = \{x^\mu : \phi > 0\}, \quad V^- = \{x^\mu : \phi < 0\}, \quad \Sigma = \{x^\mu : \phi = 0\}, \quad (14)$$

with

$$\phi = r - R(t), \quad (15)$$

where $R(t)$ is an arbitrary function that describes the history of the boundary. Then, for any of the metric coefficients, which is C^0 across the boundary and C^3 in the regions V^\pm can be written in the form [26]

$$f(t, r) = f^+(t, r)H(\phi) + f^-(t, r)[1 - H(\phi)], \quad (16)$$

where $f = \{\psi, \chi, W, \omega\}$, $H(\phi)$ is the Heavside function defined as ²

²It should be noted that the exact value of $H(\phi)$ at the point $\phi = 0$ is not uniquely defined and can be given any value.

$$H(\phi) = \begin{cases} 1, & \phi \geq 0, \\ 0, & \phi < 0, \end{cases} \quad (17)$$

and f^+ (f^-) is the function defined in the region V^+ (V^-), with the C^0 condition

$$\lim_{r \rightarrow R^+} f^+(t, r) = \lim_{r \rightarrow R^-} f^-(t, r). \quad (18)$$

Using the distribution theory, it can be shown that

$$\begin{aligned} f_{,t}(t, r) &= f_{,t}^+(t, r)H(\phi) + f_{,t}^-(t, r)[1 - H(\phi)], \\ f_{,r}(t, r) &= f_{,r}^+(t, r)H(\phi) + f_{,r}^-(t, r)[1 - H(\phi)], \\ f_{,tt}(t, r) &= f_{,tt}^+(t, r)H(\phi) + f_{,tt}^-(t, r)[1 - H(\phi)] + \dot{R}(t)^2[f_{,r}]^-\delta(\phi), \\ f_{,tr}(t, r) &= f_{,tr}^+(t, r)H(\phi) + f_{,tr}^-(t, r)[1 - H(\phi)] - \dot{R}(t)[f_{,r}]^-\delta(\phi), \\ f_{,rr}(t, r) &= f_{,rr}^+(t, r)H(\phi) + f_{,rr}^-(t, r)[1 - H(\phi)] + [f_{,r}]^-\delta(\phi), \end{aligned} \quad (19)$$

where an over-dot denotes the ordinary differentiation with respect to t , $\delta(\phi)$ the Dirac delta function, and

$$[f_{,r}]^- \equiv \lim_{r \rightarrow R(t)^+} \left(\frac{\partial f^+(t, r)}{\partial r} \right) - \lim_{r \rightarrow R(t)^-} \left(\frac{\partial f^-(t, r)}{\partial r} \right). \quad (20)$$

It should be noted that in deriving Eq.(19) we have used the relation

$$[f_{,t}]^- = -\dot{R}(t)[f_{,r}]^-. \quad (21)$$

Substituting Eq.(19) into Eq.(A.3), we find that the Einstein tensor in general can be written in the form

$$G_{\mu\nu} = G_{\mu\nu}^+ H(\phi) + G_{\mu\nu}^- [1 - H(\phi)] + \gamma_{\mu\nu} \delta(\phi), \quad (22)$$

where $G_{\mu\nu}^+$ ($G_{\mu\nu}^-$) is the Einstein tensor calculated in the region V^+ (V^-), and $\gamma_{\mu\nu}$ is the Einstein tensor calculated on the hypersurface $r = R(t)$. The non-vanishing components of $\gamma_{\mu\nu}$ in the present case are given by

$$\begin{aligned} \gamma_{00} &= -\frac{[W_r]^-}{W_0} + \omega_0^2 W_0^2 e^{-2\chi_0} \left\{ (1 - \dot{R}^2)[\chi_{,r}]^- - \frac{[\omega_{,r}]^-}{\omega_0} \right\}, \\ \gamma_{01} &= \dot{R} \left\{ \frac{[W_r]^-}{W_0} + \frac{1}{2} \omega_0 W_0^2 e^{-2\chi_0} [\omega_{,r}]^- \right\}, \\ \gamma_{03} &= \frac{1}{2} \omega_0 W_0^2 e^{-2\chi_0} \left\{ 2(1 - \dot{R}^2)[\chi_{,r}]^- - \frac{[\omega_{,r}]^-}{\omega_0} \right\}, \\ \gamma_{11} &= -\dot{R}^2 \frac{[W_{,r}]^-}{W_0}, \\ \gamma_{13} &= \frac{1}{2} \dot{R} W_0^2 e^{-2\chi_0} [\omega_{,r}]^-, \\ \gamma_{22} &= (1 - \dot{R}^2) e^{2(\psi_0 - \chi_0)} \left\{ [\chi_{,r}]^- - 2[\psi_{,r}]^- + \frac{[W_{,r}]^-}{W_0} \right\}, \\ \gamma_{33} &= (1 - \dot{R}^2) W_0^2 e^{-2\chi_0} [\chi_{,r}]^-, \quad (r = R(t)), \end{aligned} \quad (23)$$

where the quantities with the subscript “0” denote the ones calculated on the hypersurface $r = R(t)$, for example, $\omega_0 \equiv \omega(t, R(t))$, and so on. Writing the energy-momentum tensor (EMT) in a form of Eq.(22), we find that the Einstein field equations $G_{\mu\nu} = kT_{\mu\nu}$, where k is the Einstein constant, can be written as

$$G_{\mu\nu}^+ = kT_{\mu\nu}^+, \quad (r > R(t)), \quad (24)$$

$$G_{\mu\nu}^- = kT_{\mu\nu}^-, \quad (r < R(t)), \quad (25)$$

$$\gamma_{\mu\nu} = k\tau_{\mu\nu}, \quad (r = R(t)), \quad (26)$$

where $\tau_{\mu\nu}$ can be interpreted as representing the surface energy-momentum tensor, which in the present case takes the form

$$\tau_{\mu\nu} = \eta u_\mu u_\nu + p_z z_\mu z_\nu + p_\varphi \varphi_\mu \varphi_\nu + q(u_\mu \varphi_\nu + u_\nu \varphi_\mu), \quad (27)$$

where

$$\begin{aligned} \eta &\equiv -(1 - \dot{R}^2) e^{2(\psi_0 - \chi_0)} \frac{[W, r]^-}{W_0}, \\ q &\equiv -\frac{1}{2} (1 - \dot{R}^2)^{1/2} W_0 e^{2\psi_0 - 3\chi_0} [\omega, r]^-, \\ p_z &\equiv (1 - \dot{R}^2) e^{2(\psi_0 - \chi_0)} \left\{ [\chi, r]^- - 2[\psi, r]^- + \frac{[W, r]^-}{W_0} \right\}, \\ p_\varphi &\equiv (1 - \dot{R}^2) e^{2(\psi_0 - \chi_0)} [\chi, r]^-, \quad (r = R(t)), \end{aligned} \quad (28)$$

and

$$\begin{aligned} u_\mu &= (1 - \dot{R}^2)^{-1/2} e^{\chi_0 - \psi_0} \left\{ \delta_\mu^t - \dot{R} \delta_\mu^r \right\}, \\ \eta_\mu &= (1 - \dot{R}^2)^{-1/2} e^{\chi_0 - \psi_0} \left\{ \delta_\mu^r - \dot{R} \delta_\mu^t \right\}, \\ z_\mu &= e^{\psi_0} \delta_\mu^z, \quad \varphi_\mu = W_0 e^{-\psi_0} \left\{ \omega_0 \delta_\mu^t + \delta_\mu^\varphi \right\}, \end{aligned} \quad (29)$$

with the properties

$$\begin{aligned} u_\lambda u^\lambda &= -z_\lambda z^\lambda = -\varphi_\lambda \varphi^\lambda = 1, \\ u_\lambda z^\lambda &= -u_\lambda \varphi^\lambda = -z_\lambda \varphi^\lambda = 0. \end{aligned} \quad (30)$$

In order to have the physical interpretation for each term appearing in Eq.(27), we need to cast the surface EMT in its canonical form [27,28], that is, we need to solve the eigenvalue problem,

$$\tau_\nu^\mu \xi^\nu = \lambda \xi^\mu. \quad (31)$$

This system of equations will possess nontrivial solutions only when the determinant $\det|\tau_\nu^\mu - \lambda \delta_\nu^\mu| = 0$, which in the present case can be written as

$$\lambda(p_z - \lambda) [\lambda^2 - (\eta - p_\varphi)\lambda + q^2 - \eta p_\varphi] = 0. \quad (32)$$

Clearly, the above equation has four roots, $\lambda = 0$, p_z , λ_\pm , where

$$\lambda_\pm = \frac{1}{2} \left[(\eta - p_\varphi) \pm D^{1/2} \right], \quad D \equiv (\eta + p_\varphi)^2 - 4q^2. \quad (33)$$

It can be shown that the eigenvalue $\lambda = 0$ corresponds to the eigenvector $\xi_1^\mu = \eta^\mu$, where η^μ is the normal vector to the hypersurface $r = R(t)$, and given by Eq.(29). The eigenvalue $\lambda = p_z$ corresponds to the eigenvector $\xi_2^\mu = z^\mu$, which represents the pressure of the shell in the z -direction.

On the other hand, substituting Eq.(33) into Eq.(31), we find that the corresponding eigenvectors are given, respectively, by

$$\xi_\pm^\mu = (\lambda_\pm + p_\varphi) u^\mu + q \varphi^\mu. \quad (34)$$

To further study the physical meaning of λ_\pm , it is found convenient to distinguish the three cases: (a) $D > 0$; (b) $D = 0$; and (c) $D < 0$.

Case (a): In this case, the two roots λ_\pm and the two corresponding eigenvectors ξ_\pm^μ are all real and satisfy the relations,

$$\begin{aligned} (\lambda_+ + p_\varphi)(\lambda_- + p_\varphi) &= q^2, \\ \frac{\xi_\pm^\mu \xi_\pm^\nu g_{\mu\nu}}{D^{1/2}(\lambda_\pm + p_\varphi)} &= \pm 1, \\ \xi_+^\mu \xi_-^\nu g_{\mu\nu} &= 0. \end{aligned} \quad (35)$$

From these expressions we can see that when $\lambda_+ + p_\varphi > 0$, the eigenvector ξ_+^μ is timelike, while ξ_-^μ is spacelike. Setting

$$\begin{aligned}
E_{(0)}^\mu &\equiv \frac{\xi_+^\mu}{[D^{1/2}(\lambda_+ + p_\varphi)]^{1/2}}, \\
E_{(3)}^\mu &\equiv \frac{\xi_-^\mu}{[D^{1/2}(\lambda_- + p_\varphi)]^{1/2}}, \quad (\lambda_+ + p_\varphi > 0),
\end{aligned} \tag{36}$$

we find that $E_{(a)}^\mu$, ($a = 0, 1, 2, 3$) form an orthogonal base, i.e., $E_{(a)}^\lambda E_{(b)\lambda} = \eta_{ab}$, where

$$E_{(1)}^\mu = \eta^\mu, \quad E_{(2)}^\mu = z^\mu. \tag{37}$$

Then, in terms of these unit vectors, the surface EMT given by Eq.(27) takes the form

$$\tau^{\mu\nu} = \Sigma E_{(0)}^\mu E_{(0)}^\nu + p_z E_{(3)}^\mu E_{(3)}^\nu + p_{(3)} E_{(3)}^\mu E_{(3)}^\nu, \tag{38}$$

where

$$\begin{aligned}
\Sigma &\equiv \frac{D(\lambda_+ + p_\varphi)}{2q^2} \left\{ D^{1/2} p_\varphi - [p_\varphi(\eta + p_\varphi) - 2q^2] \right\}, \\
p_{(3)} &\equiv \frac{D(\lambda_- + p_\varphi)}{2q^2} \left\{ D^{1/2} p_\varphi + [p_\varphi(\eta + p_\varphi) - 2q^2] \right\}, \quad (\lambda_+ + p_\varphi > 0).
\end{aligned} \tag{39}$$

Hence, in terms of its tetrad components, $\tau^{\mu\nu}$ can be cast in the form,

$$[\tau_{(a)(b)}] = \begin{bmatrix} \Sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_z & 0 \\ 0 & 0 & 0 & p_{(3)} \end{bmatrix}. \tag{40}$$

This corresponds to the Type I fluid defined in [27]. Thus, in this case the surface EMT represents a fluid with its surface energy density given by Σ , measured by observers whose four-velocity are given by $E_{(0)}^\mu$, and the principal pressures in the directions $E_{(2)}^\mu$ and $E_{(3)}^\mu$, given respectively by p_z and $p_{(3)}$.

When $\lambda_+ + p_\varphi < 0$, the eigenvector ξ_+^μ is spacelike, while ξ_-^μ is timelike. Now if we define the two unit vectors $E_{(0)}$ and $E_{(3)}$ as

$$\begin{aligned}
E_{(0)}^\mu &\equiv \frac{\xi_-^\mu}{[D^{1/2}|\lambda_- + p_\varphi|]^{1/2}}, \\
E_{(3)}^\mu &\equiv \frac{\xi_+^\mu}{[D^{1/2}|\lambda_+ + p_\varphi|]^{1/2}}, \quad (\lambda_+ + p_\varphi < 0),
\end{aligned} \tag{41}$$

we find that the surface EMT also takes the form of Eq.(40), but now with

$$\begin{aligned}
\Sigma &\equiv \frac{D|\lambda_- + p_\varphi|}{2q^2} \left\{ D^{1/2} p_\varphi + [p_\varphi(\eta + p_\varphi) - 2q^2] \right\}, \\
p_{(3)} &\equiv \frac{D|\lambda_+ + p_\varphi|}{2q^2} \left\{ D^{1/2} p_\varphi - [p_\varphi(\eta + p_\varphi) - 2q^2] \right\}, \quad (\lambda_+ + p_\varphi < 0).
\end{aligned} \tag{42}$$

Case (b): In this case we have

$$q = \pm \frac{1}{2}(\eta - p_\varphi). \tag{43}$$

Then, the two roots λ_\pm degenerate into one. It can be shown that this multiple root corresponds to two independent eigenvectors,

$$\xi_\pm^\mu = \frac{u^\mu \pm \varphi^\mu}{\sqrt{2}}, \tag{44}$$

which are all null. From these two null vectors we can construct two unit vectors, one is timelike and the other is spacelike. But, these two unit vectors are exactly u^μ and φ^μ . Then, in the base $E_{(a)}^\mu = \{u^\mu, \eta^\mu, z^\mu, \varphi^\mu\}$, the surface EMT will take the form

$$[\tau_{(a)(b)}] = \begin{bmatrix} \Omega + \Sigma & 0 & 0 & \pm\Omega \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_z & 0 \\ \pm\Omega & 0 & 0 & \Omega - \Sigma \end{bmatrix}, \quad (45)$$

with

$$\Sigma \equiv \frac{1}{2}(\eta - p_\varphi), \quad \Omega \equiv \frac{1}{2}(\eta + p_\varphi). \quad (46)$$

This corresponds to the Type II fluid defined in [27].

Case (c): In this case the two roots λ_\pm are complex, and satisfy the relations $\lambda_- = \bar{\lambda}_+$. The two corresponding eigenvectors, given by Eq.(34), now are also complex. This means that in the present case the surface EMT cannot be diagonalized (by real similarity transformations), and is already in its canonical form. Thus, in the base consisting of the four orthogonal vectors, $E_{(a)}^\mu = \{u^\mu, \eta^\mu, z^\mu, \varphi^\mu\}$, it takes the form,

$$[\tau_{(a)(b)}] = \begin{bmatrix} \eta & 0 & 0 & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_z & 0 \\ q & 0 & 0 & p_\varphi \end{bmatrix}, \quad (47)$$

from which we can see that now η denotes the surface energy density of the shell, measured by observers whose four-velocity are given by u^μ , p_z and p_φ the principal pressures in the directions, z^μ and φ^μ , respectively, and q the heat flow in the φ^μ -direction.

It should be noted that all the above physical interpretations are valid, provided that the surface EMT satisfies some energy conditions [27].

IV. STATIONARILY ROTATING SHELLS

As an application of the formulas developed in the last section, in this section we shall consider stationarily rotating cylindrical shells. It should be noted that such shells were studied previously by various authors [29]. However, an important difference in the present case is that now we allow the spacetime inside the shell be rotating,

$$ds_-^2 = dt^2 - dr^2 - r^2(d\varphi + \Omega dt)^2 - dz^2, \quad (48)$$

where Ω is a constant that represents the angular velocity of the uniformly rotating coordinate system [30]. Note that in the above metric the axis $r = 0$ is well-defined and free of any kind of singularities and sources. When $r > \Omega^{-1}$, the metric coefficient g_{00}^- becomes negative and the Killing vector $\xi_{(0)}^\mu = \delta_0^\mu$ becomes space-like. In the following we shall assume that the above metric is valid only for $r < \Omega^{-1}$.

If we further assume that outside of the shell, the spacetime is vacuum, then it should be described by the most general vacuum Lewis solutions of the Einstein field equations, given by [7]

$$ds_+^2 = f d\bar{t}^2 + 2k d\bar{t} d\bar{\varphi} - h(d\bar{r}^2 + d\bar{z}^2) - l d\bar{\varphi}^2, \quad (49)$$

where

$$\begin{aligned} f &= a\bar{r}^{1-n} - \frac{c^2}{an^2}\bar{r}^{1+n}, \quad k = Af, \quad h = \bar{r}^{(n^2-1)/2}, \\ l &= \frac{\bar{r}^2}{f} - A^2 f, \quad A = \frac{c\bar{r}^{1+n}}{anf} + b, \end{aligned} \quad (50)$$

with a , b , c and n being constants that can be real or complex. When they are complex, certain relations have to be satisfied among them, in order to have the metric coefficients be real [16,31]. For the present purpose, we shall consider only the case where they are all real.

It should be noted that the above expressions are valid only for $n \neq 0$. However, the solutions for $n = 0$ can be obtained from them by first letting $c = n\bar{c}$ and then taking the limit $n \rightarrow 0$. Since this process is straightforward, in the following we shall consider only the solutions given by Eqs.(49) and (50), and consider the ones for $n = 0$ as their particular case. The physical meaning of the parameters a , b , c and n were first studied by Lewis [16], and more

recently by da Silva *et al.* [31]. When all the parameters are real, it can be shown that the spacetime is asymptotically flat as $\bar{r} \rightarrow +\infty$, and singular on the hypersurface $\bar{r} = 0$. This can be seen, for example, from the Kretschmann scalar,

$$R_{\mu\nu\beta\gamma}R^{\mu\nu\beta\gamma} = \frac{(n^2 + 3)(n^2 - 1)^2}{4\bar{r}^{n^2+3}}. \quad (51)$$

Thus, the Kretschmann scalar diverges as $\bar{r} \rightarrow 0$, except for the cases $n = \pm 1$. In the last two cases, it can be shown that the spacetimes are flat and belong to Petrov type O . When $n = 0, 3$, the spacetimes are Petrov type D , while all the rest are Petrov type I .

In order to apply the formulas developed in the last section, we need first to write the Lewis solutions in the form of Eq.(1). To this end, we make the following coordinate transformations

$$\begin{aligned} \bar{t} &= \left\{ \frac{\bar{\alpha}}{\sqrt{a}} \left(1 - \frac{bc}{n} \right) - \frac{b\gamma\delta}{R_0^{(1+n)/2}} \right\} t - \frac{b\gamma}{R_0^{(1+n)/2}} \varphi, \\ \bar{\varphi} &= \left\{ \frac{\bar{\alpha}c}{n\sqrt{a}} + \frac{\gamma\delta}{R_0^{(1+n)/2}} \right\} t + \frac{\gamma}{R_0^{(1+n)/2}} \varphi, \\ \bar{r} &= R_0(r + d)^{4/(1+n)^2}, \\ \bar{z} &= \frac{\beta z}{R_0^{(n^2-1)/4}}, \quad (n \neq -1), \end{aligned} \quad (52)$$

for $n \neq -1$, where $\bar{\alpha}$, β , γ , δ and d are arbitrary constants, and $R_0 \equiv [\bar{\alpha}(1+n)^2/4]^{4/(1+n)^2}$. Then, the Lewis solutions takes the form,

$$ds_+^2 = \alpha^2 A^{4\sigma}(r)(dt^2 - dr^2) - \beta^2 A^{4\sigma(2\sigma-1)}(r)dz^2 - \frac{\gamma^2}{a} A^{2(1-2\sigma)}(r)(d\varphi + \delta dt)^2, \quad (\sigma \neq 1/2), \quad (53)$$

where

$$A(r) \equiv (r + d)^{1/(2\sigma-1)^2}, \quad \alpha \equiv \bar{\alpha}R_0^{2\sigma}, \quad \sigma \equiv \frac{1-n}{4}. \quad (54)$$

From the above expressions we can see that the range of \bar{r} , $\bar{r} \in [0, \infty)$, is mapped into the range $r \in [-d, \infty)$. Hence, when $r \rightarrow +\infty$, the spacetime is asymptotically flat, and when $r \rightarrow -d$ it is singular, except for the case $n = 1$ where the spacetime is flat. As we shall see below, we will use the above metric as describing the spacetime outside of a stationary shell. Thus, to prevent spacetime singularity from happening outside the shell, we shall require that $r + d > 0$.

When $\sigma = 1/2$ (or $n = -1$), the coordinate transformations

$$\begin{aligned} \bar{t} &= \left\{ \frac{\bar{\alpha}}{\sqrt{a}}(1 + bc) - b\gamma\delta \right\} t - b\gamma\varphi, \\ \bar{\varphi} &= - \left\{ \frac{\bar{\alpha}c}{\sqrt{a}} - \gamma\delta \right\} t + \gamma\varphi, \\ \bar{r} &= e^{\alpha(r+d)}, \\ \bar{z} &= z, \quad (\sigma = 1/2), \end{aligned} \quad (55)$$

bring the metric (49) to the form

$$ds_+^2 = \alpha^2 e^{2\alpha(r+d)}(dt^2 - dr^2) - dz^2 - \frac{\gamma^2}{a}(d\varphi + \delta dt)^2, \quad (\sigma = 1/2). \quad (56)$$

As shown above, this metric represents a flat spacetime but in a rotating coordinate system.

It should be noted that the above coordinate transformations of Eqs.(52) and (55) are admissible only locally, *if both sets of the coordinates are considered as representing cylindrical coordinates*. Otherwise, it will give rise to new topology [21,32]. However, in this paper we shall take the point of view that only the coordinates $\{x^\mu\} = \{t, r, z, \varphi\}$ represent the usual cylindrical coordinates. Then, the topological identifications discussed in [21,32] are not applicable to the present case, so that the above coordinate transformations are admissible even globally.

Assuming that a stationary shell located on the hypersurface $r = r_0 > 0$ generates the spacetime described by the metric Eq.(53) or Eq.(56), and that the spacetime inside the shell is vacuum and described by the metric Eq.(48), where r_0 is a constant, we find that the first junction conditions that the metric are continuous across the surface, i.e.,

$$g_{\mu\nu}^-(r \rightarrow r_0^-) = g_{\mu\nu}^+(r \rightarrow r_0^+), \quad (57)$$

lead to

$$\alpha = A^{-2\sigma}(r_0), \quad \beta = A^{-2\sigma(2\sigma-1)}(r_0), \quad \gamma = \sqrt{a}r_0 A^{2\sigma-1}(r_0), \quad \delta = \Omega, \quad (\sigma \neq 1/2), \quad (58)$$

for $\sigma \neq 1/2$, and

$$\alpha e^{\alpha(r_0+d)} = 1, \quad \gamma = \sqrt{a}r_0, \quad \delta = \Omega, \quad (\sigma = 1/2), \quad (59)$$

for $\sigma = 1/2$. Substituting Eqs.(58) and (59), respectively, into Eq.(53) and Eq.(56), we find that

$$ds_+^2 = B^{4\sigma}(r)(dt^2 - dr^2) - B^{4\sigma(2\sigma-1)}(r)dz^2 - r_0^2 B^{2(1-2\sigma)}(r)(d\varphi + \Omega dt)^2, \quad (\sigma \neq 1/2), \quad (60)$$

for $\sigma \neq 1/2$, where

$$B(r) \equiv \left(\frac{r+d}{r_0+d} \right)^{1/(2\sigma-1)^2}, \quad (\sigma \neq 1/2), \quad (61)$$

and

$$ds_+^2 = e^{2\alpha(r-r_0)}(dt^2 - dr^2) - dz^2 - r_0^2(d\varphi + \Omega dt)^2, \quad (\sigma = 1/2), \quad (62)$$

for $\sigma = 1/2$.

Now we are at the position of calculating the surface EMT of the shell. In the following, let us consider the two cases, $\sigma \neq 1/2$ and $\sigma = 1/2$, separately.

Case a) $\sigma \neq 1/2$: In this case, from Eqs.(1), (48), (61) and (20), we find that

$$\begin{aligned} [\psi, r]^- &= \frac{2\sigma}{(2\sigma-1)(r_0+d)}, \\ [\chi, r]^- &= \frac{4\sigma^2}{(2\sigma-1)^2(r_0+d)}, \\ [\omega, r]^- &= 0, \\ [W, r]^- &= -\frac{d}{r_0+d}, \quad (\sigma \neq 1/2). \end{aligned} \quad (63)$$

Substituting the above expressions into Eq.(28), we find

$$\begin{aligned} \eta &= \frac{d}{r_0(r_0+d)}, \\ q &= 0, \\ p_z &= \frac{4\sigma(1-\sigma)r_0 - (2\sigma-1)^2d}{(2\sigma-1)^2r_0(r_0+d)}, \\ p_\varphi &= \frac{4\sigma^2}{(2\sigma-1)^2(r_0+d)}, \quad (\sigma \neq 1/2). \end{aligned} \quad (64)$$

The combination of Eqs.(64) and (33) shows that in the present case we have $D > 0$. Thus, now the fluid is type *I*.

For the shell to be physically acceptable, it has to satisfy some energy conditions, weak, dominant, or strong [27]. It can be shown that if the condition

$$0 \leq \sigma \leq 1, \quad d \geq 0, \quad (65)$$

holds, the weak energy condition will be satisfied, and if the condition

$$d \geq \begin{cases} \frac{2\sigma(1-\sigma)r_0}{(2\sigma-1)^2}, & 0 \leq \sigma \leq \frac{1}{3}, \\ \frac{4\sigma^2 r_0}{(2\sigma-1)^2}, & \frac{1}{3} < \sigma \leq 1, \end{cases} \quad (66)$$

holds, the dominant energy condition will be satisfied, while if the condition

$$d \geq \begin{cases} -\frac{4\sigma^2 r_0}{(2\sigma-1)^2}, & 0 \leq \sigma \leq \frac{1}{4}, \\ -r_0, & \frac{1}{4} < \sigma \leq 1, \end{cases}. \quad (67)$$

holds, the strong energy condition will be satisfied. Clearly, for $0 \leq \sigma \leq 1$, all the three energy conditions can be satisfied by properly choosing the free parameter d . It is interesting to note that in the case $\Omega = 0$ it was shown that the solutions have physics only when $0 \leq \sigma \leq 1/2$ [8]. Recently we extended it to the range $\sigma \in [0, 1]$ [5], which is exactly the same as that obtained above for the case with rotation, $\Omega \neq 0$.

Case b) $\sigma = 1/2$: In this case, Combining Eqs.(1), (48), (62) and (20), we find that

$$[\psi, r]^- = 0, \quad [\chi, r]^- = \alpha, \quad [\omega, r]^- = 0, \quad [W, r]^- = -1, \quad (\sigma = 1/2). \quad (68)$$

Then, the surface EMT of the shell is given by Eq.(27) with

$$\begin{aligned} \eta &= \frac{1}{r_0}, \\ q &= 0, \\ p_z &= \frac{\alpha r_0 - 1}{r_0}, \\ p_\varphi &= \alpha, \quad (\sigma = 1/2). \end{aligned} \quad (69)$$

It can be shown that when $\alpha \geq 0$ the weak and strong energy conditions will be satisfied, while when $0 \leq \alpha \leq 1/r_0$ the dominant energy condition will be satisfied. Similar to the last case, the fluid now is also type I .

V. CONCLUSIONS

In this paper, the main properties of cylindrical spacetimes with rotation were studied by using the NP formulas. The physical interpretations of each component of the Riemann tensor was given by considering null geodesic deviations. It would be very interesting to see their experimental implications.

In Sec. *III* the spacetimes were extended to include rotating cylindrical shells, and the general expression of the surface energy-momentum tensor of the shells were given in terms of the discontinuation of the first derivatives of the metric coefficients. This would be very useful in studying gravitational collapse of a rotating cylindrical shell. As a matter of fact, this was one of our main motivations for such a study. As an application of the formulas developed, we considered a stationary shell that generates the Lewis solutions, which represent the most general vacuum solutions of the cylindrical spacetimes with rotation [7], by assuming that inside the shell the spacetime is flat. It was shown that by properly choosing one of the free parameters, the shell can satisfy all the three energy conditions, provided $0 \leq \sigma \leq 1$, where σ is related to the mass per unit length of the cylindrical shell [8,31]. This range is exactly the same as that obtained in the static case [5].

APPENDIX A: THE CHRISTOFFEL SYMBOLS AND THE EINSTEIN TENSOR

Corresponding to the metric (1), the non-vanishing Christoffel symbols, defined by

$$\Gamma_{\nu\lambda}^\mu = \frac{1}{2}g^{\mu\sigma} \{g_{\sigma\lambda,\nu} + g_{\nu\sigma,\lambda} - g_{\nu\lambda,\sigma}\}, \quad (A.1)$$

are given by

$$\begin{aligned} \Gamma_{00}^0 &= \chi_{,t} - \psi_{,t} - \omega^2 W^2 e^{-2\chi} \left(\psi_{,t} - \frac{W_{,t}}{W} \right), \quad \Gamma_{01}^0 = \chi_{,r} - \psi_{,r} - \frac{1}{2}\omega W^2 e^{-2\chi} \omega_{,r}, \\ \Gamma_{03}^0 &= -\omega W^2 e^{-2\chi} \left(\psi_{,t} - \frac{W_{,t}}{W} \right), \quad \Gamma_{11}^0 = \chi_{,t} - \psi_{,t}, \quad \Gamma_{13}^0 = -\frac{1}{2}W^2 e^{-2\chi} \omega_{,r}, \end{aligned}$$

$$\begin{aligned}
\Gamma_{22}^0 &= e^{2(2\psi-\chi)}\psi_{,t}, \quad \Gamma_{33}^0 = -W^2 e^{-2\chi} \left(\psi_{,t} - \frac{W_{,t}}{W} \right), \\
\Gamma_{00}^1 &= \chi_{,r} - \psi_{,r} + \omega^2 W^2 e^{-2\chi} \left(\psi_{,r} - \frac{W_{,r}}{W} - \frac{\omega_{,r}}{\omega} \right), \quad \Gamma_{01}^1 = \chi_{,t} - \psi_{,t}, \\
\Gamma_{03}^1 &= \frac{1}{2} \omega W^2 e^{-2\chi} \left(2\psi_{,r} - 2\frac{W_{,r}}{W} - \frac{\omega_{,r}}{\omega} \right), \quad \Gamma_{11}^1 = \chi_{,r} - \psi_{,r}, \\
\Gamma_{22}^1 &= -e^{2(2\psi-\chi)}\psi_{,r}, \quad \Gamma_{33}^1 = W^2 e^{-2\chi} \left(\psi_{,r} - \frac{W_{,r}}{W} \right), \quad \Gamma_{02}^2 = \psi_{,t}, \quad \Gamma_{12}^2 = \psi_{,r}, \\
\Gamma_{00}^3 &= -\omega \left(\chi_{,t} + \psi_{,t} - 2\frac{W_{,t}}{W} - \frac{\omega_{,r}}{\omega} \right) + \frac{1}{2} \omega^2 W^2 e^{-2\chi} \omega_{,r}, \\
\Gamma_{03}^3 &= -(1 - \omega^2 W^2 e^{-2\chi}) \left[\psi_{,t} - \frac{W_{,t}}{W} \right], \quad \Gamma_{11}^3 = -\omega(\chi_{,t} - \psi_{,t}), \\
\Gamma_{13}^3 &= \frac{W_{,r}}{W} - \psi_{,r} + \frac{1}{2} \omega W^2 e^{-2\chi} \omega_{,r}, \quad \Gamma_{22}^3 = -\omega e^{2(2\psi-\chi)} \psi_{,t}, \\
\Gamma_{33}^3 &= \omega W^2 e^{-2\chi} \left[\psi_{,t} - \frac{W_{,t}}{W} \right],
\end{aligned} \tag{A.2}$$

while the non-vanishing components of the Einstein tensor are given by

$$\begin{aligned}
G_{00} &= - \left\{ \frac{W_{,rr}}{W} - \frac{1}{W} (\chi_{,t} W_{,t} + \psi_{,r} W_{,r}) + \psi_{,t}^2 + \psi_{,r}^2 - \frac{3}{4} \omega^2 W^4 e^{-4\chi} \omega_{,r}^2 \right. \\
&\quad \left. + \omega^2 W^2 e^{-2\chi} \left[\chi_{,tt} - \chi_{,rr} + \frac{\omega_{,rr}}{\omega} + \psi_{,t}^2 - \psi_{,r}^2 - 2\frac{\chi_{,r}\omega_{,r}}{\omega} + 3\frac{\omega_{,r}W_{,r}}{\omega W} + \frac{1}{4} \left(\frac{\omega_{,r}}{\omega} \right)^2 \right] \right\}, \\
G_{01} &= - \left\{ \frac{W_{,tr}}{W} - \frac{1}{W} (\chi_{,t} W_{,r} + \chi_{,r} W_{,t}) + 2\psi_{,t}\psi_{,r} \right. \\
&\quad \left. + \frac{1}{2} \omega W^2 e^{-2\chi} \left[\omega_{,tr} - 2\chi_{,t}\omega_{,r} + 3\omega_{,r} \frac{W_{,t}}{W} \right] \right\}, \\
G_{03} &= -\omega W^2 e^{-2\chi} \left\{ \chi_{,tt} - \chi_{,rr} + \frac{\omega_{,rr}}{2\omega} + \psi_{,t}^2 - \psi_{,r}^2 - \frac{\chi_{,r}\omega_{,r}}{\omega} + \frac{3\omega_{,r}W_{,r}}{2\omega W} + \frac{3}{4} W^2 e^{-2\chi} \omega_{,r}^2 \right\}, \\
G_{11} &= - \left\{ \frac{W_{,tt}}{W} + \psi_{,t}^2 + \psi_{,r}^2 - \frac{1}{W} (\chi_{,t} W_{,t} + \chi_{,r} W_{,r}) - \frac{1}{4} W^2 e^{-2\chi} \omega_{,r}^2 \right\}, \\
G_{13} &= -\frac{1}{2} W^2 e^{-2\chi} \left\{ \omega_{,tr} + 3\omega_{,r} \frac{W_{,t}}{W} - 2\chi_{,t}\omega_{,r} \right\}, \\
G_{22} &= e^{2(2\psi-\chi)} \left\{ 2(\psi_{,tt} - \psi_{,rr}) - (\chi_{,tt} - \chi_{,rr}) - \frac{1}{W} (W_{,tt} - W_{,rr}) \right. \\
&\quad \left. - (\psi_{,t}^2 - \psi_{,r}^2) + \frac{2}{W} (\psi_{,t} W_{,t} - \psi_{,r} W_{,r}) - \frac{1}{4} W^2 e^{-2\chi} \omega_{,r}^2 \right\}, \\
G_{33} &= -W^2 e^{-2\chi} \left\{ \chi_{,tt} - \chi_{,rr} + \psi_{,t}^2 - \psi_{,r}^2 + \frac{3}{4} W^2 e^{-2\chi} \omega_{,r}^2 \right\},
\end{aligned} \tag{A.3}$$

where $\chi_{,r} \equiv \partial\chi/\partial r$ and $\chi_{,t} \equiv \partial\chi/\partial t$, etc.

APPENDIX B: THE SPIN COEFFICIENTS AND THE RICCI AND WEYL SCALARS

Choosing the null tetrad as given by Eq.(2), we find that the spin coefficients are given by

$$\begin{aligned}
\rho &= l_{\mu;\nu} m^\mu \bar{m}^\nu = -\frac{e^{\psi-\chi}}{2\sqrt{2}AW} (W_{,t} + W_{,r}), \\
\kappa &= l_{\mu;\nu} m^\mu l^\nu = 0 \\
\sigma &= l_{\mu;\nu} m^\mu m^\nu = -\frac{e^{\psi-\chi}}{2\sqrt{2}A} \left[2(\psi_{,t} + \psi_{,r}) - \frac{W_{,t} + W_{,r}}{W} \right],
\end{aligned}$$

$$\begin{aligned}
\tau &= l_{\mu;\nu} m^\mu n^\nu = -i \frac{W e^{\psi-2\chi}}{2\sqrt{2}} \omega_{,r}, \\
\alpha &= \frac{1}{2} (l_{\mu;\nu} n^\mu \bar{m}^\nu - m_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu) = i \frac{W e^{\psi-2\chi}}{4\sqrt{2}} \omega_{,r}, \\
\epsilon &= \frac{1}{2} (l_{\mu;\nu} n^\mu l^\nu - m_{\mu;\nu} \bar{m}^\mu l^\nu) = \frac{e^{\psi-\chi}}{2\sqrt{2}A} \left[\chi_{,t} + \chi_{,r} - (\psi_{,t} + \psi_{,r}) - \frac{A_{,t} + A_{,r}}{A} \right], \\
\mu &= -n_{\mu;\nu} \bar{m}^\mu m^\nu = \frac{A e^{\psi-\chi}}{2\sqrt{2}W} (W_{,t} - W_{,r}), \\
\nu &= -n_{\mu;\nu} \bar{m}^\mu n^\nu = 0, \\
\lambda &= -n_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu = \frac{A e^{\psi-\chi}}{2\sqrt{2}} \left[2(\psi_{,t} - \psi_{,r}) - \frac{W_{,t} - W_{,r}}{W} \right], \\
\pi &= -n_{\mu;\nu} \bar{m}^\mu l^\nu = i \frac{W e^{\psi-2\chi}}{2\sqrt{2}} \omega_{,r}, \\
\beta &= \frac{1}{2} (l_{\mu;\nu} n^\mu m^\nu - m_{\mu;\nu} \bar{m}^\mu m^\nu) = -i \frac{W e^{\psi-2\chi}}{4\sqrt{2}} \omega_{,r}, \\
\gamma &= \frac{1}{2} (l_{\mu;\nu} n^\mu n^\nu - m_{\mu;\nu} \bar{m}^\mu n^\nu) = -\frac{A e^{\psi-\chi}}{2\sqrt{2}} \left[\chi_{,t} - \chi_{,r} - (\psi_{,t} - \psi_{,r}) + \frac{A_{,t} - A_{,r}}{A} \right].
\end{aligned} \tag{B.1}$$

Then, the corresponding Ricci and Weyl scalars are given, respectively, by

$$\begin{aligned}
\Phi_{00} &= \frac{1}{2} S_{\mu\nu} l^\mu l^\nu \\
&= -\frac{e^{2(\psi-\chi)}}{4A^2W} [W_{,tt} + 2W_{,tr} + W_{,rr} - 2(\chi_{,t} + \chi_{,r})(W_{,t} + W_{,r}) + 2W(\psi_{,t} + \psi_{,r})^2], \\
\Phi_{01} &= \frac{1}{2} S_{\mu\nu} l^\mu m^\nu \\
&= -i \frac{W e^{2\psi-3\chi}}{8A} \left[\omega_{,rr} + \omega_{,tr} - 2\omega_{,r}(\chi_{,t} + \chi_{,r}) + \frac{3\omega_{,r}}{W}(W_{,t} + W_{,r}) \right], \\
\Phi_{02} &= \frac{1}{2} S_{\mu\nu} m^\mu m^\nu \\
&= \frac{1}{2} e^{2(\psi-\chi)} \left[\psi_{,tt} - \psi_{,rr} + \frac{1}{W}(\psi_{,t}W_{,t} - \psi_{,r}W_{,r}) - \frac{1}{2W}(W_{,tt} - W_{,rr}) + \frac{1}{4}W^2 e^{-2\chi} \omega_{,r}^2 \right], \\
\Phi_{11} &= \frac{1}{4} S_{\mu\nu} (l^\mu n^\nu + m^\mu \bar{m}^\nu) \\
&= \frac{1}{4} e^{2(\psi-\chi)} \left[\psi_{,tt} - \psi_{,rr} - \psi_{,t}^2 + \psi_{,r}^2 + \frac{1}{W}(\psi_{,t}W_{,t} - \psi_{,r}W_{,r}) + \chi_{,rr} - \chi_{,tt} - \frac{3}{4}W^2 e^{-2\chi} \omega_{,r}^2 \right], \\
\Phi_{12} &= \frac{1}{2} S_{\mu\nu} n^\mu m^\nu \\
&= -i \frac{1}{8} A W e^{2\psi-3\chi} \left[\omega_{,rr} - \omega_{,tr} + 2\omega_{,r}(\chi_{,t} - \chi_{,r}) - \frac{3\omega_{,r}}{W}(W_{,t} - W_{,r}) \right], \\
\Phi_{22} &= \frac{1}{2} S_{\mu\nu} n^\mu n^\nu \\
&= -\frac{A^2 e^{2(\psi-\chi)}}{4W} [W_{,tt} - 2W_{,tr} + W_{,rr} - 2(\chi_{,t} - \chi_{,r})(W_{,t} - W_{,r}) + 2W(\psi_{,t} - \psi_{,r})^2], \\
\Lambda &= \frac{1}{24} R \\
&= -\frac{1}{12} e^{2(\psi-\chi)} \left[\psi_{,tt} - \psi_{,rr} - \psi_{,t}^2 + \psi_{,r}^2 + \frac{1}{W}(\psi_{,t}W_{,t} - \psi_{,r}W_{,r}) - (\chi_{,tt} - \chi_{,rr}) \right. \\
&\quad \left. - \frac{1}{W}(W_{,tt} - W_{,rr})n - \frac{1}{4}W^2 e^{-2\chi} \omega_{,r}^2 \right],
\end{aligned} \tag{B.2}$$

and

$$\begin{aligned}
\Psi_0 &= -C_{\mu\nu\lambda\delta} l^\mu m^\nu l^\lambda m^\delta \\
&= -\frac{e^{2(\psi-\chi)}}{2A^2} \left\{ \psi_{,tt} + 2\psi_{,tr} + \psi_{,rr} + 2(\psi_{,t} + \psi_{,r})^2 - 2(\psi_{,t} + \psi_{,r})(\chi_{,t} + \chi_{,r}) \right. \\
&\quad \left. - \frac{1}{2W} [W_{,tt} + 2W_{,tr} + W_{,rr} - 2(W_{,t} + W_{,r})(\chi_{,t} + \chi_{,r})] \right\}, \\
\Psi_1 &= -C_{\mu\nu\lambda\delta} l^\mu n^\nu l^\lambda m^\delta \\
&= -i \frac{W e^{2\psi-3\chi}}{8A} \left[\omega_{,rr} + \omega_{,tr} - 2\omega_{,r}(\psi_{,t} + \psi_{,r} + \chi_{,t} + \chi_{,r}) + \frac{3\omega_{,r}}{W}(W_{,t} + W_{,r}) \right], \\
\Psi_2 &= -\frac{1}{2} C_{\mu\nu\lambda\delta} [l^\mu n^\nu l^\lambda n^\delta - l^\mu n^\nu m^\lambda \bar{m}^\delta] \\
&= \frac{1}{6} e^{2(\psi-\chi)} \left\{ \psi_{,tt} - \psi_{,rr} - \chi_{,tt} + \chi_{,rr} + 2(\psi_{,t}^2 - \psi_{,r}^2) \right. \\
&\quad \left. + \frac{1}{2W} [W_{,tt} - W_{,rr} - 4(\psi_{,t} W_{,t} - \psi_{,r} W_{,r})] - W^2 e^{-2\chi} \omega_{,r}^2 \right\}, \\
\Psi_3 &= -C_{\mu\nu\lambda\delta} n^\mu l^\nu n^\lambda \bar{m}^\delta \\
&= i \frac{1}{8} A W e^{2\psi-3\chi} \left[\omega_{,rr} - \omega_{,tr} + 2\omega_{,r}(\psi_{,t} - \psi_{,r} + \chi_{,t} - \chi_{,r}) - \frac{3\omega_{,r}}{W}(W_{,t} - W_{,r}) \right], \\
\Psi_4 &= -C_{\mu\nu\lambda\delta} n^\mu \bar{m}^\nu n^\lambda \bar{m}^\delta \\
&= -\frac{1}{2} A^2 e^{2(\psi-\chi)} \left\{ \psi_{,tt} - 2\psi_{,tr} + \psi_{,rr} + 2(\psi_{,t} - \psi_{,r})^2 - 2(\psi_{,t} - \psi_{,r})(\chi_{,t} - \chi_{,r}) \right. \\
&\quad \left. - \frac{1}{2W} [W_{,tt} - 2W_{,tr} + W_{,rr} - 2(W_{,t} - W_{,r})(\chi_{,t} - \chi_{,r})] \right\}, \tag{B.3}
\end{aligned}$$

where

$$S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R, \tag{B.4}$$

and $C_{\mu\nu\sigma\delta}$ denotes the Weyl tensor. In terms of the Riemann and Ricci tensors, it is given by

$$\begin{aligned}
C_{\mu\nu\lambda\delta} &= R_{\mu\nu\lambda\delta} - \frac{1}{2} (g_{\mu\lambda} R_{\nu\delta} + g_{\nu\delta} R_{\mu\lambda} - g_{\nu\lambda} R_{\mu\delta} - g_{\mu\delta} R_{\nu\lambda}) \\
&\quad - \frac{1}{6} (g_{\mu\delta} g_{\nu\lambda} - g_{\mu\lambda} g_{\nu\delta}) R. \tag{B.5}
\end{aligned}$$

In terms of the Weyl and Ricci scalars, the Weyl and Ricci tensors are given, respectively, by

$$\begin{aligned}
C_{\mu\nu\lambda\delta} &= -4 \left\{ (\Psi_2 + \bar{\Psi}_2) (l_{[\mu} n_{\nu]} l_{[\lambda} n_{\delta]} + m_{[\mu} \bar{m}_{\nu]} m_{[\lambda} \bar{m}_{\delta]}) \right. \\
&\quad - (\Psi_2 - \bar{\Psi}_2) (l_{[\mu} n_{\nu]} m_{[\lambda} \bar{m}_{\delta]} + m_{[\mu} \bar{m}_{\nu]} l_{[\lambda} n_{\delta]}) \\
&\quad + \Psi_0 n_{[\mu} \bar{m}_{\nu]} n_{[\lambda} \bar{m}_{\delta]} \\
&\quad + \Psi_1 (l_{[\mu} n_{\nu]} n_{[\lambda} \bar{m}_{\delta]} + n_{[\mu} \bar{m}_{\nu]} l_{[\lambda} n_{\delta]} + n_{[\mu} \bar{m}_{\nu]} \bar{m}_{[\lambda} m_{\delta]} + \bar{m}_{[\mu} m_{\nu]} n_{[\lambda} \bar{m}_{\delta]}) \\
&\quad - \Psi_2 (l_{[\mu} m_{\nu]} n_{[\lambda} \bar{m}_{\delta]} + n_{[\mu} \bar{m}_{\nu]} l_{[\lambda} m_{\delta]}) \\
&\quad - \Psi_3 (l_{[\mu} n_{\nu]} l_{[\lambda} m_{\delta]} + l_{[\mu} m_{\nu]} l_{[\lambda} n_{\delta]} - l_{[\mu} m_{\nu]} m_{[\lambda} \bar{m}_{\delta]} - m_{[\mu} \bar{m}_{\nu]} l_{[\lambda} m_{\delta]}) \\
&\quad + \Psi_4 l_{[\mu} m_{\nu]} l_{[\lambda} m_{\delta]} \\
&\quad + \bar{\Psi}_0 n_{[\mu} m_{\nu]} n_{[\lambda} m_{\delta]} \\
&\quad + \bar{\Psi}_1 (l_{[\mu} n_{\nu]} n_{[\lambda} m_{\delta]} + n_{[\mu} m_{\nu]} l_{[\lambda} n_{\delta]} + n_{[\mu} m_{\nu]} m_{[\lambda} \bar{m}_{\delta]} + m_{[\mu} \bar{m}_{\nu]} n_{[\lambda} m_{\delta]}) \\
&\quad - \bar{\Psi}_2 (l_{[\mu} \bar{m}_{\nu]} n_{[\lambda} m_{\delta]} + n_{[\mu} m_{\nu]} l_{[\lambda} \bar{m}_{\delta]}) \\
&\quad - \bar{\Psi}_3 (l_{[\mu} n_{\nu]} l_{[\lambda} \bar{m}_{\delta]} + l_{[\mu} \bar{m}_{\nu]} l_{[\lambda} n_{\delta]} - l_{[\mu} \bar{m}_{\nu]} \bar{m}_{[\lambda} m_{\delta]} - \bar{m}_{[\mu} m_{\nu]} l_{[\lambda} \bar{m}_{\delta]}) \\
&\quad \left. + \Psi_4 l_{[\mu} \bar{m}_{\nu]} l_{[\lambda} \bar{m}_{\delta]} \right\}, \tag{B.6}
\end{aligned}$$

and

$$\begin{aligned}
R_{\mu\nu} = & 2 \{ \Phi_{00} n_\mu n_\nu + \Phi_{22} l_\mu l_\nu \\
& - \Phi_{01} (n_\mu \bar{m}_\nu + n_\nu \bar{m}_\mu) - \bar{\Phi}_{01} (n_\mu m_\nu + n_\nu m_\mu) \\
& + \Phi_{02} \bar{m}_\mu \bar{m}_\nu + \bar{\Phi}_{02} m_\nu m_\mu \\
& (2\Phi_{11} - 3\Lambda) (l_\mu n_\nu + l_\nu n_\mu) + (2\Phi_{11} + 3\Lambda) (m_\mu \bar{m}_\nu + m_\nu \bar{m}_\mu) \\
& - \Phi_{12} (l_\mu \bar{m}_\nu + l_\nu \bar{m}_\mu) - \bar{\Phi}_{12} (l_\mu m_\nu + l_\nu m_\mu) \} .
\end{aligned} \tag{B.7}$$

It should be noted that Eqs.(B.6) and (B.7) hold for the general case.

ACKNOWLEDGMENT

We would like to thank P.S. Letelier, N.O. Santos, and M.F.A. da Silva for useful discussions. The financial assistance from CNPq and FAPERJ (AW) is gratefully acknowledged.

-
- [1] P.S. Joshi, *Global Aspects in Gravitation and Cosmology* (Clarendon, Oxford, 1993).
 - [2] M.W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993); C. Gundlach, Adv. Theor. Math. Phys. **2**, 1 (1998); *gr - qc/971208*.
 - [3] S.L. Shapiro and S.A. Teukolsky, Phys. Rev. Lett. **66**, 994 (1991).
 - [4] T.A. Apostolatos and K.A. Thorne, Phys. Rev. **D46**, 2433 (1992).
 - [5] A.Z. Wang, M.F.A. da Silva, and N.O. Santos, Class. Quantum Grav. **14**, 2417 (1997).
 - [6] M.F.A. da Silva, A.Z. Wang, F.M. Paiva and N.O. Santos, Phys. Rev. **D61**, 044003 (2000).
 - [7] D. Kramer, H. Stephani, E. Herlt, and M. MacCallum, *Exact Solutions of Einstein's Field Equations*, (Cambridge University Press, Cambridge, England, 1980), p.221.
 - [8] W.B. Bonnor, Gen. Rel. Grav. **24** 551 (1992); W.B. Bonnor, J.B. Griffiths, and M.A.H. MacCallum, *ibid.* **26**, 687 (1994).
 - [9] E. Newmann and R. Penrose, J. Math. Phys. **3**, 566 (1962); *ibid.* **4**, 998 (1963).
 - [10] W. Israel, Nuovo Cimento, **B44**, 1 (1966); *ibid.* **48B**, 463(E) (1967).
 - [11] G. Darmon, *Mémoires des Sciences Mathématiques* (Gauthier-Villars, Paris), Fasc. 25.
 - [12] A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Électromagnétisme* (Masson, Paris), p.61.
 - [13] A. Papapetrou and A. Hamoui, Ann. Inst. Henri. Poincaré, **9**, 179 (1968).
 - [14] A.H. Taub, J. Math. Phys. **21**, 1423 (1980).
 - [15] W.B. Bonnor and P.A. Vickers, Gen. Relativ. Grav. **13**, 29 (1981).
 - [16] T. Lewis, Proc. Roy. Soc. Lond. **A136**, 176 (1932).
 - [17] B. Mashhoon and H. Quevedo, Phys. Lett. **A151**, 464 (1990); H. Quevedo and B. Mashhoon, "Cylindrically Symmetric Gravitational Waves with Rotational Properties", in Proc. SILARG VII (Mexico City, Dec. 1990).
 - [18] K. Thorne, Phys. Rev. **B138**, 251 (1965).
 - [19] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (W.H. Freeman and Company, San Francisco, 1973) pp.953 - 955.
 - [20] P.R.C.T. Pereira, N. O. Santos, and A.Z. Wang, Class Quantum Grav. **13**, 1641 (1996).
 - [21] M.A.H. MacCallum and N.O. Santos, Class. Quantum Grav. **15**, 1627 (1998).
 - [22] P. Szekeres, J. Math. Phys. **6**, 1387 (1965); *ibid.* **7**, 751 (1966).
 - [23] A.Z. Wang, Phys. Rev. **D44**, 1120 (1991).
 - [24] P. S. Letelier and A.Z. Wang, Phys Rev. **D49**, 5105 (1994); *ibid.*, **D51**, 5968 (E) (1995).
 - [25] V.P. Frolov, in "Problems in the General Theory of Relativity and Theory of Group Representations," edited by N.G. Basov, Proceedings (Trudy) of the P.N. Lebedev Physics Institute, Vol. 96, translated from Russian by A. Mason (Consultants Bureau New York and London, 1979) pp.73 - 185.
 - [26] A.Z. Wang, Phys. Rev. **D45**, 3534 (1992); J. Math. Phys. **33**, 1065 (1992).
 - [27] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime*, (Cambridge University Press, Cambridge, 1973), pp.88-96.
 - [28] P. S. Letelier and A.Z. Wang, "Cosmic Bubbles and Rotating Black Holes," in *Gravitation: The spacetime Structure*, SILARG VIII, Proceedings of the 8th Latin American Symposium on Relativity and Gravitation, Aguas de Lindóia, Brazil, 25-30 July, 1993, Edited by P.S. Letelier and W.A. Rodrigues, Jr. (World Scientific, 1994), pp. 332-338.
 - [29] E. Frehland, Commun. Math. Phys. **26**, 307 (1972); A. Papapetrou, A. Macedo, and M.M. Som, Inter. J. Theor. Phys. **17**, 975 (1978); S.R. Jordan and J.D. McGrea, J. Phys. A: Math. Gen. **15**, 1807 (1982); J. Stachel, J. Math. Phys. **25**, 338 (1984).
 - [30] L.D. Landau and E.M. Lifshitz, *The Classical Theory of fields*, (Pergamon Press, New York, 1975), p.254.

- [31] M.F.A. da Silva, L. Herrera, F.M. Paiva, and N.O. Santos, Gen. Relativ. Grav. **27**, 859 (1995); Class. Quantum Grav. **12**, 111 (1995).
- [32] M.A.H. MacCallum, Gen. Relativ. Grav. **30**, 131 (1998).

FIGURE CAPTIONS

Fig.1 S_O and S_P are two infinitesimal 2-elements spanned by e_2 and e_3 and orthogonal to the null geodesic C defined by l^μ , passing S_O and S_P at the points O and P , respectively. A light beam meets S_O in the circle S .

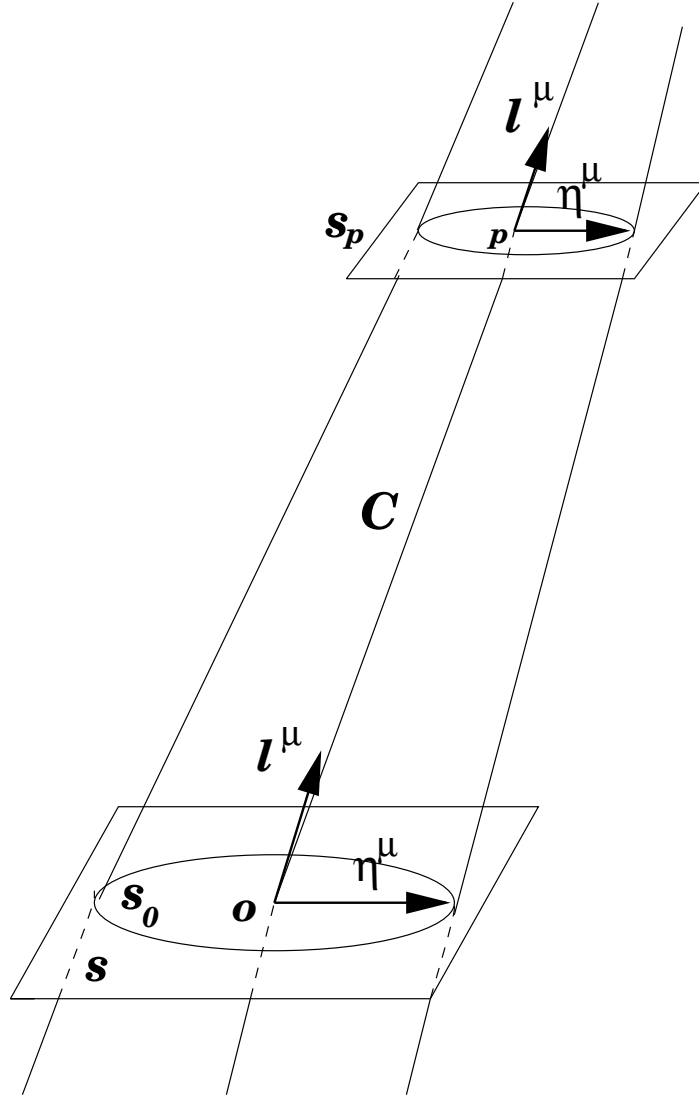


Fig. 1

Fig.2 (a) The image of the circle S on S_P is contracted because of the interaction of Φ_{00} . (b) The image of the circle S on S_P is deflected into an ellipse with its main major axis along e_2 because of the interaction of Ψ_0 .

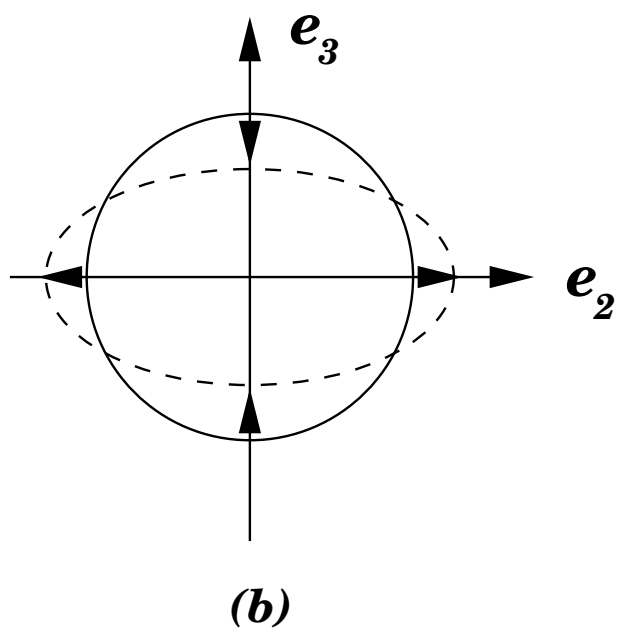
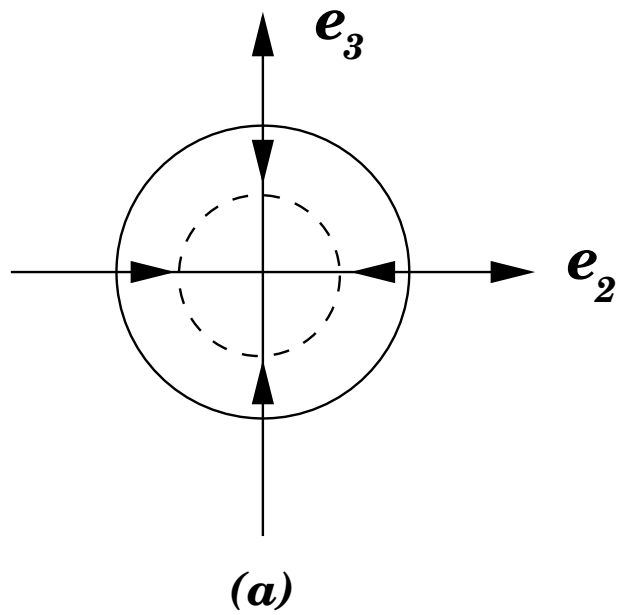


Fig. 2

Fig.3 A spherical ball consisting of photons cuts S_O in the circle S with the point O as its center. The image of the ball at the point P is turned into a spheroid with the main major axis along a line at 45° with respect to e_1 in the plane spanned by e_1 and e_3 because of the interaction of Ψ_1 and Φ_{01} , while the rays are left undeflected in the e_2 -direction.

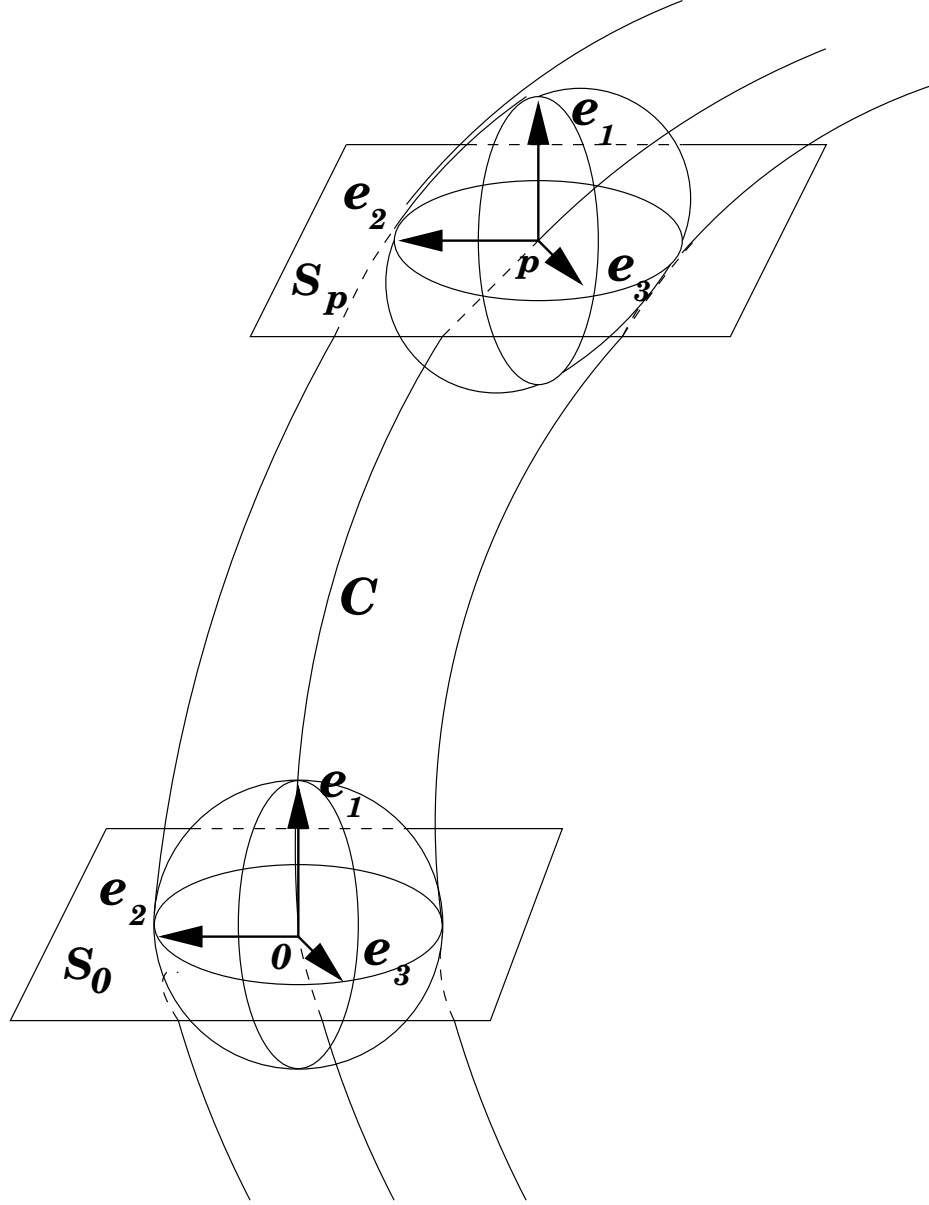


Fig. 3